Hyperbolic Localization and Weight Functor

$\lambda \in X_\cdot(T)$, $\quad S_\lambda = N_F t^\lambda G_O / G_O$

$\quad = \{ x \in Gr : \lim_{a \to 0} 2\rho^\vee(a) \cdot x = t^\lambda \}$

$T_\lambda = N_F^- t^\lambda G_O / G_O$

$\quad = \{ x \in Gr : \lim_{a \to \infty} 2\rho^\vee(a) \cdot x = t^\lambda \}$

Lemma $\quad T_\lambda \cap S_\mu \neq \emptyset \Rightarrow \mu \geq \lambda$, $\quad T_\lambda \cap S_\lambda = \{ t^\lambda \}$.

$\hat{\mathfrak{g}} = \mathfrak{g}_F \oplus \mathbb{C}K \oplus \mathbb{C}d \qquad \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$

$\Lambda_0 \in \hat{\mathfrak{h}}^\vee \quad \mathfrak{h} \oplus \mathbb{C}d \mapsto 0, \; K \mapsto 1$

$V(\Lambda_0)$: highest weight representation

$\mathfrak{g}_O \oplus \mathbb{C}d$ stablize $l \subset V(\Lambda_0)$

$\Psi : Gr = G_F / G_O \hookrightarrow \mathbb{P}(V(\Lambda_0))$

$\quad g \mapsto [gl]$

Claim : $\Psi(t^\lambda) \in \mathbb{P}(V_{-\iota(\lambda)}) \qquad \iota : X_\cdot(T) \to X^\cdot(T)$

$\Rightarrow \Psi(\overline{S_\lambda}) \subseteq \mathbb{P}(\bigoplus_{\chi \geq -\iota(\lambda)} V_\chi)$

$\quad \Psi(\overline{T_\lambda}) \subseteq \mathbb{P}(\bigoplus_{\chi \leq -\iota(\lambda)} V_\chi)$

$T_\lambda \cap S_\mu \neq \emptyset \Rightarrow -\iota(\lambda) \geq -\iota(\mu) \Rightarrow \mu \geq \lambda$

Prop $\quad \dim \overline{Gr_\lambda} \cap S_\mu = \underline{\langle \rho, \lambda + \mu \rangle} \qquad \lambda \in X_\cdot(T)^+$.

Pf. $\quad \overline{Gr_\lambda} \cap S_\mu \neq \emptyset \Rightarrow t^\mu \in \overline{Gr_\lambda} \Rightarrow w_0 \lambda \leq \mu \leq \lambda$.

Use induction

① $\mu = w_0 \lambda$, $\quad \overline{Gr_\lambda} \subset \overline{T_{w_0\lambda}} \Rightarrow \overline{Gr_\lambda} \cap S_\mu = \{ t^{w_0 \lambda} \}$

② $\mu = \lambda$, $\quad \overline{Gr_\lambda} \subset \overline{S_\lambda}$, $\quad S_\lambda \cap \overline{Gr_\lambda} \overset{\text{open}}{\subset} \overline{Gr_\lambda}$

$$\mathcal{S}_{<\lambda} = \bar{\mathcal{S}}_\lambda \cap \Theta \qquad \Theta = t^\lambda (G_r^o)^c.$$

Braden's hyperbolic localization theorem

$\mathbb{G}_m \curvearrowright X$ (normal) $\qquad F = X^{\mathbb{G}_m}$

$$X^+ = \{ x \in X : \lim_{a \to 0} a \cdot x \text{ exists} \}$$

$$X^- = \{ x \in X : \lim_{a \to \infty} a \cdot x \text{ exists} \}$$

$$\begin{tikzcd} & F \arrow[dl, hook, "f^+"'] \arrow[dr, "f^-"] & \\ X^+ \arrow[dr, "g^+"'] & & X^- \arrow[dl, "g^-"] \\ & X & \end{tikzcd}$$

$\mathcal{S} \in D(X)$ weakly equivariant $\qquad \rho : \mathbb{G}_m \times X \to X$

$\rho^* \mathcal{S} = L \boxtimes \mathcal{S}$

$$\begin{aligned} &\mathcal{S}^{*!} = (f^-)^* (g^-)^! \mathcal{S} \\ &\downarrow \\ &\mathcal{S}^{!*} = (f^+)^! (g^+)^* \mathcal{S} \qquad \text{is isomorphism} \end{aligned}$$

adjoint $\mathcal{S} \to (g^+)_* (g^+)^* \mathcal{S}$

apply $(-)^{*!}$ $\quad \mathcal{S}^{*!} \to \underbrace{(f^-)^* (g^-)^! (g^+)_*}_{(f^-)_* (f^+)^!} (g^+)^* \mathcal{S} = (f^+)^! (g^+)^* = \mathcal{S}^{!*}$

$j : X \setminus X^+ \to X$, $\quad j_! j^* \mathcal{S} \to \mathcal{S} \to (g^+)_* (g^+)^* \mathcal{S} \xrightarrow{+1}$

suffice to show $(j_! j^* \mathcal{S})^{*!} = 0$

i.e. $\mathcal{S}^{*!} = \mathcal{S}^{!*}$ if $\mathcal{S}|_{X^+} = 0$.

normal : X can be covered by subspace of $\mathbb{A}^N$, where $\mathbb{G}_m$ acts linearly.

suffice to show when $X = \mathbb{A}^N$.

weak equivalence : Lemma

Y base space $W \to Y$ $\mathbb{G}_m$-equivariant vector bundle.

$W = W_+ \oplus W_-$,

$E = \mathbb{P}(W) \setminus \mathbb{P}(W_+)$, $B = \mathbb{P}(W_-)$

$p : [w_+, w_-] \mapsto [w_-]$

$$\begin{tikzcd} E \arrow[r, shift left, "p"] & B \arrow[l, shift left, "i"] \arrow[r, "\phi"] & Y \end{tikzcd}$$

$p \circ i = \mathrm{id}$

$\phi_* p_* S = \phi_* i^* S$, S weakly equivariant

$\phi_* p_! S = \phi_* i^! S$

$\mu : \mathbb{G}_m \times E \to E$,

graph $\Gamma_\mu \subset \mathbb{G}_m \times E \times E$. pts: (a, e, ae)

$\Gamma = \overline{\Gamma_\mu} \subset \mathbb{A}^1 \times \bar{E} \times E$. pts: (a, e, ae), $a \neq 0$

$\tilde{\Gamma} = \bar{\Gamma} \subset \mathbb{A}^1 \times E \times \mathbb{P}(W)$ $(0, e, \lim_{a \to 0} ae) \in \{0\} \times E \times B$.

$\tilde{\Gamma} = \Gamma \Rightarrow pr_1 : \Gamma \to \mathbb{A}^1 \times E$ proper. $pr_2 : \Gamma \to \mathbb{A}^1 \times E$.

$\phi_* p_* \curvearrowright j_! j^* S \to S \to i_* i^* S \xrightarrow{+1}$

suffice: $\phi_* p_* S = 0$ $S|_B = 0$.

$\psi : \mathbb{A}^1 \times E \xrightarrow{\mathrm{id} \times \phi p} \mathbb{A}^1 \times Y$ goal: $\psi_*(k_{\mathbb{A}^1} \boxtimes S) = 0$.

$\psi_*(k_{\mathbb{A}^1} \boxtimes S) \to \psi_* pr_{2*} pr_2^*(k_{\mathbb{A}^1} \boxtimes S)$

$= \psi_* pr_{1!} pr_2^*(k_{\mathbb{A}^1} \boxtimes S) = \tilde{S}$

Claim $\tilde{S} = k_{\mathbb{G}_m} \boxtimes \phi_* p_* S \to \psi_*(k_{\mathbb{A}^1} \boxtimes S)$

Omitted

$X = \mathbb{A}^N = V_0 \times V_+ \times V_-$

$F = V_0$,

$X^+ = V_0 \times V_+$

$X^- = V_0 \times V_-$

$\bar{X} = V_0 \times (\mathbb{P}(V_+ \oplus V_- \oplus k) \setminus \mathbb{P}(V_-))$

$\overline{X^+} = V_0 \times \mathbb{P}(V_+ \oplus k)$ $\overline{X^-} = X^-$

$b : X \to \bar{X}$, $h : \bar{X} \setminus X^- \to \bar{X}$, $\pi : \bar{X} \to V_0$

$(bg^-)_* (bg^-)^! b_! S \to b_! S \to h_* h^* b_! S \xrightarrow{+1}$

apply $\bar\pi$: $(\bar\pi b g^-)_*(g^-)^! S = (f^-)^*(g^-)^! S = S^{*!}$

suffice: $\bar\pi_* b_! S = 0$, $\bar\pi_* h_* h^* b_! S = 0$.

① Lemma:

$$\begin{tikzcd} \overline{X} \arrow[d, shift right, "p"'] \\ \overline{X^+} \arrow[u, shift right, "i"'] \arrow[d, "\phi"'] \\ V_0 \end{tikzcd}$$

$\bar\pi_* b_! S$

$= \phi_* i^* b_! S$

$S|_{X^+} = 0$, $(b_! S)|_{\overline{X}\setminus X} = 0$

$\Rightarrow i^* b_! S = 0$

② Lemma $E = \overline{X} \setminus X^- = V_0 \times (\mathbb{P}(V_+ \oplus V_- \oplus k) \setminus \mathbb{P}(V_- \oplus k))$

$$\begin{tikzcd} E \arrow[d, shift right, "p"'] \arrow[rrr, hook, "h"] & & & \overline{X} \arrow[dlll, bend left, "\bar\pi"] \\ B = \overline{X}^+ \setminus X^+ = V_0 \times \mathbb{P}(V_+) \arrow[u, shift right, "i"'] \arrow[d, "\phi"'] \arrow[rrru, hook, "a"] & & & \\ V_0 & & & \end{tikzcd}$$

$\underline{\bar\pi_* h_*} h^* b_! S$

$= \phi_* i^* h^* b_! S$

$= \phi_* a^* b_! S$

$S|_{X^+} = 0$, $(b_! S)|_{\overline{X}\setminus X} = 0$

$\Rightarrow a^* b_! S = 0$

Cor $f^\pm : F \rightleftarrows X^\pm$ ($\pi^\pm$ the map back)

$(f^\pm)^* = (\pi^\pm)_*$, $(f^\pm)^! = (\pi^\pm)_!$

Thm weak equivalence

$\underline{H^k_{T_\mu}(Gr, A) \xrightarrow{\sim} H^k_c(S_\mu, A)}$ $\quad A = IC_\lambda$.

$X = \overline{Gr_\lambda}$, $\quad X^-_\mu = T_\mu \cap \overline{Gr_\lambda}$, $\quad X^+_\mu = S_\mu \cap \overline{Gr_\lambda}$.

$$(f^-)^*(g^-)^! A \xrightarrow{\sim} (f^+)^!(g^+)^* A \qquad F = \{t^\mu \in \overline{Gr_\lambda}\}$$

$$= (\pi_-)_*(g^-)^! A \qquad \overset{\shortparallel}{(\pi_+)_!(g^+)^* A}$$

Prop $\quad H^k_c(S_\mu, A)$ is zero if $k \neq \langle 2\rho, \mu\rangle$.

$\overline{Gr_\lambda} \cap S_\mu$ covers S_μ.

$\dim \overline{Gr_\lambda} \cap S_\mu = \langle \rho, \lambda + \mu\rangle$

$A \in Perv(Gr) \qquad i^* A \in D^{\leq \dim}$

$\Rightarrow$ zero if $k > \langle 2\rho, \mu\rangle$

同理, $H^k_{T_\mu}(Gr, A) = 0$ if $k < \langle 2\rho, \mu\rangle$. □

weight functor $F_\mu(A) = H_c^{\langle 2\rho, \mu\rangle}(S_\mu, A)$.

Last: $H^*(Gr, A) = \bigoplus_{\mu \in X_*(T)} F_\mu(A)$.

$$H_c^{\langle 2\rho, \mu\rangle}(S_\mu, A) = H_c^{\langle 2\rho, \mu\rangle}(\overline{S_\mu}, A)$$

& $\quad H_c^{\langle 2\rho, \lambda\rangle}(\overline{S_\mu}, A) = H_c^{\langle 2\rho, \lambda\rangle}(S_\lambda, A) \quad \lambda \leq \mu$.

$$H_c^{\langle 2\rho, \mu\rangle + \text{odd}}(\overline{S_\mu}, A) = 0$$

Pf. Induction.

$$\begin{tikzcd}
 & H^{\langle 2\rho,\mu\rangle}(Gr, A) \arrow[drr] & & \\
H^{\langle 2\rho,\mu\rangle}_{\overline{T}_\mu}(Gr, A) \arrow[ur] \arrow[dr, "\wr"'] & & & H^{\langle 2\rho,\mu\rangle}_c(\overline{S}_\mu, A) \\
 & H^{\langle 2\rho,\mu\rangle}_{T_\mu}(Gr, A) \arrow[r, "\sim"] & H^{\langle 2\rho,\mu\rangle}_c(S_\mu, A) \arrow[ur, "\wr"'] &
\end{tikzcd}$$

$\square$